

Dynamics of globally coupled oscillators: progress and perspectives

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In this paper we discuss recent progress in research of ensembles of mean field coupled oscillators. Without an ambition to present a comprehensive review, we outline most interesting from our viewpoint results and surprises, as well as interrelations between different approaches.

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Studies of large systems of interacting oscillatory elements is a popular and extensively developing branch of nonlinear science. The number of publications on the subject grows rapidly, with many crucial contributions published in the Chaos journal. In addition to purely academic interest, this research finds promising applications in various fields. Representative examples are understanding of pedestrian synchrony on footbridges and of other social phenomena, development of efficient high-frequency power sources, modeling and control of neuronal rhythms, etc. In this paper we present our view of the recent development of this highly interesting field.

I. INTRODUCTION

In the second half of the 17th century Engelbert Kaempfer, Dutch physician, made a journey to South-East Asia and later published a book, describing his trip¹. In particular, in his memoirs he gives an account of a spectacular show, which happens if a swarm of fireflies occupies a tree: the insects “hide their Lights all at once, and a moment after make it appear again with the utmost regularity and exactness”. This phenomenon of *self-synchronization in a large population of interacting oscillatory objects* not only remains an appealing entertainment — be it an excursion on a night river in Thailand to observe fireflies or cycling in a large group of people, equipped with electronic “bikeflies”, as a part of a festival in Chicago — but it stays in the focus of scientific interest within many decades. The studies of various aspects of the collective dynamics in large oscillatory networks attract attention of physicists and applied mathematicians and find applications ranging from electrochemistry to quantum electronics, and from bridge engineering to social sciences.

A particularly popular and expanding field of applications is neuroscience. For the first time the link between synchronous flashing of the fireflies and origin of the brain rhythms was established, on a quantitative level, by the famous mathematician Norbert Wiener in his monograph on Cybernetics², in the chapter entitled “Brain waves and self-organizing systems”, see also Ref. [3]. Putting forward the hypothesis that the brain waves emerge due to “phenomenon of the pulling together of frequencies”, he questioned, whether the same nonlinear mechanism takes place in case of fireflies, crickets, and other species exhibiting

collective oscillatory behavior, and suggested that experiments on fireflies and on electronic systems can shed light on the brain wave dynamics.

Since that many experiments have been reported, including those with electrochemical^{4,5} and electronic oscillators⁶, metronomes⁷, Josephson junctions⁸, laser arrays⁹, yeast cells¹⁰, and gene-manipulated clocks in bacteria¹¹. They are complimented by observations of many social phenomena like pedestrian synchrony on footbridges¹², synchronous hand clapping in opera houses¹³, egg-laying in bird colonies¹⁴, and menstrual synchrony¹⁵ (the latter effect remains controversial¹⁶). This research was also accompanied by an essential progress in the theoretical description, that we outline below, along with the open questions.

II. SOLVABLE MODELS

A few years after publication of the Wiener's book, Arthur Winfree presented a first mathematical description of collective synchrony in a large population of biological oscillators¹⁷. Reducing the dynamics of each oscillator to that of only one variable, the phase φ (we will discuss conditions for this reduction below), he proposed the mean-field model

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N} \Gamma(\varphi_k) \sum_{j=1}^N I(\varphi_j) , \quad (1)$$

where $N \gg 1$ is the number of units, ω_k are their natural frequencies, and ε quantifies the strength of the interaction. The function $\Gamma(\varphi_k)$ describes the phase sensitivity of the oscillator to an infinitesimal perturbation and is typically called the phase response curve, PRC. Notice that the PRC can be experimentally obtained by repeating stimulation of an isolated system. Finally, the forcing function $I(\varphi_j)$ describes the effect of the j -th unit on the unit k . In this setup the coupling is assumed to be global, i.e. of the all-to-all type, and functions Γ , I are assumed to be identical for all interactions. Thus, the inhomogeneity of the system is due to a distribution of frequencies ω_k only. Although the model is quite complicated for the analysis, Winfree has shown that it exhibits a transition to a macroscopic synchronized state, characterized by non-zero mean field $N^{-1} \sum_j I(\varphi_j)$. He discovered that the collective synchrony is a threshold phenomenon: the transition occurs if the coupling strength ε is large enough or the inhomogeneity, i.e. the width of the distribution of ω_k , is sufficiently small. For recent studies on the Winfree model see Refs. [18,19]; it has been shown that it can be treated analytically at least if functions Γ , I contain only one Fourier

harmonics.

The next pioneering step has been done by Yoshiki Kuramoto in his seminal publication 40 years ago²⁰. The model he suggested can be considered as the weak-coupling limit of Eq. (1). Indeed, if $\varepsilon \ll \omega$, then each Eq. (1) can be averaged over the oscillation period; then each term $\Gamma(\varphi_k)I(\varphi_j)$ yields a function of the phase difference, $g(\varphi_j - \varphi_k)$, so that the averaged model reads

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N} \sum_{j=1}^N g(\varphi_j - \varphi_k) . \quad (2)$$

The general case of an arbitrary 2π -periodic function g is discussed later, while the simplest case $g(\cdot) = \sin(\cdot)$ corresponds to the famous Kuramoto model (notice that choice of the sine function is not a result of some approximation but just the simplest solvable case):

$$\dot{\varphi}_k = \omega_k + \frac{\varepsilon}{N} \sum_{j=1}^N \sin(\varphi_j - \varphi_k) = \omega_k + \varepsilon R \sin(\Theta - \varphi_k) . \quad (3)$$

Here $Re^{i\Theta} = N^{-1} \sum_j e^{i\varphi_j}$ is the complex mean field, R and Θ are its amplitude and phase, respectively. Kuramoto solved the problem in the thermodynamical limit $N \rightarrow \infty$, using a self-consistent approach: assuming a harmonic mean field with unknown amplitude and frequency, he obtained closed integral equations for these two quantities. The celebrated result is the existence of the critical coupling, ε_c , proportional to the width of the frequency distribution. For sub-threshold coupling the mean field is exactly zero, while for $\varepsilon > \varepsilon_c$, a nontrivial solution with non-zero mean field appears; the amplitude of the field grows as $\sqrt{\varepsilon - \varepsilon_c}$ and its frequency equals the central frequency of the distribution of ω_k (which is assumed to be symmetric and unimodal). Thus, appearance of the collective mode can be treated as a second-order nonequilibrium phase transition. As has been shown later, the Kuramoto model with the uniform frequency distribution exhibits a jump of the order parameter²¹.

III. COLLECTIVE DYNAMICS OF THE KURAMOTO MODEL

The results of Kuramoto gave an enormous impact on the development of the field, with still an increasing number of publications on the subject. The model (3) and its extension due to Sakaguchi and Kuramoto²², who accounted for a possible phase shift in coupling,

$$\dot{\varphi}_k = \omega_k + \varepsilon R \sin(\Theta - \varphi_k + \alpha) , \quad (4)$$

became a paradigmatic model for analysis of large oscillator ensembles.

A. Watanabe-Strogatz theory

We now briefly discuss an interesting and important mathematical property of the Kuramoto-Sakaguchi model (4), namely its *partial integrability*. We start this discussion by consideration of the simplest case of identical units, $\omega_k = \omega$. As has been shown in the seminal publications by Watanabe and Strogatz²³ (WS), the dynamics of this system can be described by three global variables ρ , Φ , Ψ and $N - 3$ constants of motion ψ_k . Here the variable $0 \leq \rho \leq 1$ is roughly similar to the mean field amplitude R ; Φ and ξ are angular variables; often it is convenient to combine two variables as $z = \rho e^{i\Phi}$, see also Refs. [24]. The main idea of the powerful WS theory is as follows. Consider $N > 3$ identical oscillators governed by

$$\dot{\varphi}_k = \omega(t) + \text{Im} [H(t)e^{-i\varphi_k}] , \quad (5)$$

where $H(t)$ is an arbitrary common forcing. Obviously, Eq. (4) is a particular case of Eq. (5). The latter can be re-written as

$$\frac{d}{dt} (e^{i\varphi_k}) = i\omega(t)e^{i\varphi_k} + \frac{1}{2}H(t) - \frac{e^{i2\varphi_k}}{2}H^*(t) . \quad (6)$$

Next, we transform N variables φ_k to complex z , $|z| < 1$, and N real ξ_k , according to

$$e^{i\varphi_k} = \frac{z + e^{i\xi_k}}{1 + z^* e^{i\xi_k}} . \quad (7)$$

This transformation, found by WS and written in this form in Refs. [24,25], is known as the Möbius transformation. Since the system is over-determined, one requires

$$N^{-1} \sum_{k=1}^N e^{i\xi_k} = \langle e^{i\xi_k} \rangle = 0 . \quad (8)$$

Substituting Eq. (7) into Eq. (6) we obtain, after straightforward manipulations:

$$\begin{aligned} \dot{z} + \left[\dot{z}z^* - z\dot{z}^* + i\dot{\xi}_k(1 - |z|^2) \right] e^{i\xi_k} &= i\omega z + \frac{H}{2} \\ &- \frac{H^*}{2}z^2 + [i\omega(1 + |z|^2) + (z^*H - zH^*)] e^{i\xi_k} \\ &+ \left[\dot{z}^* + i\omega z^* + \frac{H}{2}z^{*2} - \frac{H^*}{2} \right] e^{2i\xi_k} . \end{aligned} \quad (9)$$

Averaging these equations over k , using Eq. (8) and $\langle \dot{\xi}_k e^{i\xi_k} \rangle = 0$, we obtain

$$\dot{z} - i\omega z - \frac{H}{2} + \frac{H^*}{2} z^2 = \left[\dot{z}^* + i\omega z^* - \frac{H^*}{2} + \frac{H}{2} z^{*2} \right] \langle e^{2i\xi_k} \rangle$$

and this equation is obviously satisfied, if

$$\dot{z} = i\omega z + \frac{H}{2} - \frac{H^*}{2} z^2 . \quad (10)$$

Substitution of Eq. (10) and its complex conjugate into Eq. (9) yields:

$$i\dot{\xi}_k(1 - |z|^2) = \left[i\omega + \frac{z^* H - z H^*}{2} \right] (1 - |z|^2) .$$

Excluding the fully synchronous case $|z| = 1$ and introducing $\xi_k = \alpha + \psi_k$, where all $\psi_k = \text{const}$, we obtain:

$$\dot{\alpha} = \omega + \text{Im}(z^* H) . \quad (11)$$

Expressions (10,11) represent the Watanabe-Strogatz equations, which completely describe the evolution of an ensemble of identical oscillators.

Extension of the WS theory for the case of non-identical units depends on the structure of the ensemble. Consider first the case of a *hierarchical population*, which consists of M groups of identical units, so that units in each group are subject to the same field²⁴. In this case the dynamics of the ensemble obeys the system of M coupled WS equations (10,11). Another practically important case is a large population, $N \rightarrow \infty$, which can be characterized by a distribution of frequencies $g(\omega)$. This system is described by WS variables $z(\omega, t)$, $\alpha(\omega, t)$ and the equations for $\partial_t z$, $\partial_t \alpha$ are similar to (10,11), see Ref. [24].

B. From WS to Ott-Antonsen theory

There exist a particular case, when the WS equations can be essentially simplified. As has been shown in Refs. [24], for large N and for the uniform distribution of constants of motion ψ_k , the WS variable z coincides with the *local* Kuramoto mean field

$$Z(\omega, t) = \int_0^{2\pi} w(\varphi, t|\omega) e^{i\varphi} d\varphi , \quad (12)$$

(where $w(\varphi, t|\omega)$ is the distribution density of oscillators' phases at frequency ω ; $\int d\varphi w(\varphi, t|\omega) = 1$) to be distinguished from the global mean field

$$Y(t) = \int_{-\infty}^{\infty} g(\omega) Z(\omega, t) d\omega . \quad (13)$$

As a result, one obtains closed equations for Z :

$$\frac{\partial Z(\omega, t)}{\partial t} = i\omega Z + \frac{H(\omega, t)}{2} - \frac{H^*(\omega, t)}{2} Z^2, \quad (14)$$

$$\frac{\partial \alpha(\omega, t)}{\partial t} = \omega + \text{Im} [Z^* H(\omega, t)] . \quad (15)$$

In the most common case of the mean field coupling $H = \varepsilon e^{i\beta} Y$, the forcing H is independent on α and Eq. (15) becomes irrelevant. Hence, we are left with Eq. (14) which also appears in the recent theory by Ott and Antonsen (OA)^{26,27}, briefly introduced below.

Consider the thermodynamical limit of the model

$$\dot{\varphi}_k = \omega_k + \text{Im} [H(t) e^{-i\varphi_k}] \quad (16)$$

and the corresponding continuity equation for the distribution of phases

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial \varphi} (w \dot{\varphi}) = 0 . \quad (17)$$

Next, let us introduce Fourier components of the density (the generalized local Daido order parameters)

$$Z_m(\omega, t) = \int_0^{2\pi} w(\omega, \varphi, t) e^{im\varphi} d\varphi, \quad (18)$$

cf. Eq. (12). Computing

$$\frac{\partial Z(\omega, t)}{\partial t} = \int_0^{2\pi} \frac{\partial w}{\partial t} e^{im\varphi} d\varphi = - \int_0^{2\pi} \frac{\partial (w \dot{\varphi})}{\partial t} e^{im\varphi} d\varphi,$$

integrating by parts and using Eq. (16), one obtains an infinite-dimensional system of ODEs:

$$\frac{\partial Z(\omega, t)}{\partial t} = im\omega Z_m + \frac{m}{2} (H Z_{m-1} - H^* Z_{m+1}) . \quad (19)$$

A particular case $Z_m = Z_1^m = Z^m$, called the OA manifold, reduces all the equations (19) to a single Eq. (14). Thus, the OA manifold corresponds to the special solution of the WS theory, with the uniform distribution of constants of motion ψ . Furthermore, OA argued that for continuous frequency distribution $g(\omega)$ the OA manifold is the only attractor²⁷ (although relaxation to it may be rather slow²⁸). A particular case of the Lorentzian distribution $g(\omega) = [\pi(\omega^2 + 1)]^{-1}$ admits a further essential simplification. Under assumption that $Z(\omega)$ is analytic in the upper half-plane, one computes the integral in Eq. (13) by virtue of residues and obtains $Y = Z(i)$. Substituting this along with $\omega = i$ into Eq. (14) one obtains the OA equation

$$\dot{Y} = \left(\frac{\varepsilon e^{i\beta}}{2} - 1 \right) Y - \frac{\varepsilon e^{-i\beta}}{2} Y^2 Y^* \quad (20)$$

for the evolution of the mean field in the Kuramoto-Sakaguchi model. Looking for a stationary synchronous solution, we set $Y = R_0 e^{i\nu t}$ and obtain the amplitude $R_0 = \sqrt{1 - 2/\varepsilon \cos \beta}$ and frequency $\nu = (\varepsilon \cos \beta - 1) \tan \beta$ of the mean field.

For an illustration of the WS and OA theories we consider the model of two interacting populations by Abrams et al.²⁹. Both populations consist of the same number of identical oscillators, but the coupling strength within the group differs from that between the groups. The dynamics can be fully described by two coupled systems of WS equations, and is therefore six-dimensional²⁴. In the particular case of uniformly distributed constants ψ , i.e. on the OA manifold, the dimension reduces to four. The latter case, studied in Ref. [29], reveals, for certain parameters, an interesting solution, when one population synchronizes while the other stays between complete synchrony and full asynchrony, i.e. $R_1 = 1$ and $0 < R_2 < 1$. This symmetry-breaking state is called *chimera*. (Notice that originally chimera states have been introduced for nonlocally coupled oscillator chains³⁰.) Interesting, that R_2 can be time-periodic. Analysis of the full six-dimensional systems exhibits additional solutions with the quasiperiodic chimera states²⁴. Theoretical predictions have been confirmed in recent experiments with two groups of metronomes, placed on platforms which were coupled via springs⁷.

C. Generalizations of the Kuramoto model

There are many studies of different generalizations of the Kuramoto model. Here we briefly mention those where the coupling function is purely harmonic like in Eq. (4), but the overall setup is more complex:

a. Multi-modal frequency distribution and several interacting populations. We have seen that an ensemble with a Lorentzian distribution of frequencies is described by the OA Eq. (20). A multi-modal distribution of frequencies can be modeled as a superposition of Lorentzians and, hence, described by a system of coupled OA Eqs. (20), see e.g. Ref. [31]. Moreover, this approach can be generalized to a set of populations with frequencies, distributed around completely different central values, whereas the latter can be either in resonance³² or not³³.

b. Nontrivial transitions for unimodal distributions. For a long time it has been assumed that in the Kuramoto-Sakaguchi model (4) with different unimodal distributions of

frequencies the dynamics of the mean field is qualitatively the same as for the Lorentzian distribution (Eq. (20)). Rather surprisingly, Omelchenko and Wolfrum³⁴ have demonstrated rather complex transition scenarios, including first-order transitions and bistability, for some unimodal distributions.

c. Complex coupling schemes. The Kuramoto-Sakaguchi model describes a “direct” coupling scheme: the mean field, calculated algebraically from the states of all oscillators, enters the equations for the phases. The coupling scheme can be, however, more complex: the mean field may act on some macroscopic variables that obey a set of generally nonlinear differential equations, and the acting force is a function of these variables. For example, in a description of pedestrian synchrony on a footbridge³⁵, one describes each pedestrian by an individual phase variable, but one needs also equations for the swinging mode of the bridge. The latter is driven by the field created by all pedestrians and, in its turn, affects their gaits. Similarly, electronic⁶ or electrochemical^{4,5} oscillators can be coupled through the common macroscopic current or voltage, which obeys macroscopic equations describing the coupling circuit. In this way one also describes synchronization of Josephson junctions³⁶ or spin-torque³⁷ oscillators.

d. Nonhomogeneous populations. There is another generalization of the standard Kuramoto-Sakaguchi coupling. The latter assumes that all the oscillators make the same contribution to the mean field and that the mean field acts on all oscillators in an equal way. Refs. [38 and 39] generalized this to the situation, where the global field still can be introduced, but oscillators contribute to it differently, i.e. with different amplitudes and phase shifts, and the field also acts differently on different oscillators. For a physical implementation one can consider a receiver which collects signals emitted from the oscillators (where the attenuation and the phase shift are due to signal propagation properties), and the governing signal is then transmitted to the oscillators, being also subject to attenuation and phase shift depending on the positions of the units³⁹. One variant of an inhomogeneity of the population is when it consists of two groups with different reaction to the force⁴⁰: some are “conformists” (they follow the force) and some are “contrarians” (they tend to be in anti-phase with the forcing). Another possibility, quite general for neural ensembles, is when some oscillators are inhibitors (trying to desynchronize others, coupling is repulsive) and others exert an excitatory action, attempting to synchronize (attracting the phase)^{41,42}.

e. Effects of noise. Independent noisy forces acting on oscillators of a population counteract synchronization. In this sense noise is a source of disorder, similar to the distribution of natural frequencies. Due to noise, synchronization is a threshold phenomenon already for identical oscillators and the transition occurs at a critical coupling which is proportional to the noise intensity. In the thermodynamic limit such a system is described analytically by a nonlinear Fokker-Planck equation which differs from the continuity equation (17) by a term $\sim \sigma^2 \frac{\partial^2 w}{\partial \varphi^2}$, where σ is the amplitude of the noise. Completely opposite is the action of a common noise: it tends to synchronize the population of oscillators and for identical units this effect can be described within the WS framework⁴³.

f. Finite-size fluctuations. Kuramoto and OA theories have been developed in the thermodynamic limit of infinitely large populations; WS theory applies to any population size, but for identical populations only. In finite populations with different natural frequencies of units one expects to observe fluctuations, both prior and beyond the synchronization transition, which is defined ambiguously in this case. In Ref. [44] the Kuramoto model with a uniform distribution of frequencies and a relatively small number of oscillators have been shown to be chaotic prior to synchronization transition, the maximal Lyapunov exponent however decreases with the system size as $\lambda \sim N^{-1}$. Above synchronization transition only regular dynamics have been observed. However, for $N \gg 1$ and close to the synchronization transitions the regime is complex: if it is not chaotic, then it is quasiperiodic with a large number of incommensurate frequencies; here statistical approaches based on finite-size scaling have been applied to find critical indices of N -dependence^{45,46}.

g. Kuramoto model on networks. Kuramoto model has been initially formulated for the ensemble of globally coupled oscillators. Recently, it has been extensively studied for other coupling configurations, i.e. for networks of different complexity, including small-world networks⁴⁷, multidimensional hypercubic lattices⁴⁶, networks with a modular structure⁴⁸, and an ensemble with an extra leading element (hub)⁴⁹. Dynamical properties of the transition to synchronization depend on the network topology⁵⁰. One of the popular applications here is study of synchronization of power grids⁵¹.

h. External forcing and collective phase resetting. As one can see from Eq. (20), the macroscopic order parameter obeys an equation for a self-sustained oscillator. Thus, the collective mode has the same properties as such an oscillator. In particular, if the ensemble is forced by a periodic force, the latter can entrain the frequency of the mean field oscilla-

tions^{26,52}. In the case of a pulse force, the latter can shift the phase of the collective mode; in this way one defines the phase response curve for the collective mode⁵³.

i. Mathematical results. The Kuramoto model has been extensively studied on the physical and computational level, but rigorous mathematical results are sparse. They mainly refer to stability analysis of the desynchronized state^{28,54}; for a description of the synchronization transition as a bifurcation problem see Refs. [55].

IV. GENERAL COUPLING FUNCTIONS

Now we come back to Eq. (2). For a general coupling function g , this equation represents the Daido model^{56,57}. Expanding g into Fourier series, $g(\eta) = \sum_n g_n e^{in\eta}$, and introducing generalized order parameters

$$Z_n = N^{-1} \sum_{j=1}^N e^{in\varphi_j}, \quad (21)$$

one can re-write the model as

$$\dot{\varphi}_k = \omega_k + \varepsilon \sum_n g_n Z_n e^{-in\varphi_k}. \quad (22)$$

One can see that generally all order parameters should be determined self-consistently, so that a complete analysis at general coupling is still missing. We mention here several interesting regimes appearing due to high harmonics in coupling function.

j. Clustering. Even for identical oscillators the WS theory does not apply, and a population can build several clusters⁵⁸, each of them consisting of fully synchronized units.

k. Heteroclinic cycles. In Ref. [59] nontrivial regimes of clustering and switching between different cluster states have been found in a population of identical oscillators with a function g containing the first and the second harmonics. This complex dynamics is due to a heteroclinic cycle in the phase space, well understood for small networks⁶⁰.

l. Multi-branch entrainment. Already for two harmonic components in the coupling function g , the r.h.s. of Eq. (22) as a function of φ can have two stable branches. This is a new situation compared to the standard Kuramoto model: now for a given mean field entrainment at two different microscopic phases is possible. This leads to an enormous multiplicity of microscopic states and to a complex structure of macroscopic regimes^{57,61,62}.

V. NON-PHASE MODELS

We started our discussion of collective ensemble dynamics from phase models. This approach relies on the well-known idea that motion along the limit cycle of an autonomous system can be parameterized by a single variable, the phase. Moreover, if the interaction of the oscillator with the environment is weak so that one can neglect the deviation of the trajectory from the cycle of the autonomous system, then the low-dimensional phase description remains valid. Generally, for strong coupling, one has to analyze full dynamical models which is a complicated problem that can hardly be treated analytically.

Numerical studies reveal that transition to collective synchrony is a quite general property, observed for various periodic, noisy, and even chaotic oscillators^{4,63}, including, e.g., spiking and bursting model neurons. The picture is, however, non-universal. The most transparent and studied model is ensemble of coupled Stuart-Landau oscillators⁶⁴ (this oscillator is the simplest prototype of a limit-cycle oscillator, with a perfect separation of amplitude and phase variables, see Eq. (24) below), and essentially new effects are the oscillation quenching, when too strong coupling effectively introduces additional damping to ensemble elements, and the collective chaos in a system of initially periodic units.

On the other hand, non-identical chaotic phase-coherent Rössler oscillators adjust their frequencies and phases (this effect is known as phase synchronization of chaos⁶⁵) and produce a nearly periodic mean field. The oscillators themselves remain chaotic, but irregular fluctuations of the amplitudes turn to be averaged out in the mean field; the small fluctuations of the latter are presumably due to the finite-size effect⁶³. Experimental studies on chaotic electrochemical oscillators⁴ confirm theoretical predictions.

Another important class are rotators: these systems are described by an angle-like variable, which is very similar to the phase, and do not have amplitudes. If an equation governing the rotator's angle is one-dimensional, the dynamics can be reduced to a Kuramoto-type phase model, what has been extensively discussed in context of Josephson junctions³⁶. Quite often the inertia of rotators cannot be neglected and, hence, they are described by a second-order equation for the angle variable. In this case the dynamics can strongly deviate from that of the Kuramoto model, e.g. transition to synchrony may exhibit hysteresis, similarly to a first-order phase transition⁶⁶. A particular subclass is constituted by globally coupled rotators without damping: this conservative system yields the so-called Hamiltonian

Mean Field (HMF) model⁶⁷; see Refs. [68] for a review of recent results and relation to the Kuramoto model.

VI. COMPLEX COLLECTIVE DYNAMICS AROUND SYNCHRONY

We have discussed in detail the main effect observed in globally coupled systems, namely emergence of the collective mode, which is well-understood in the simplest case, when many units synchronize and therefore their outputs sum up coherently, contributing to this mode. We have also mentioned, that the mode itself can exhibit chaotic dynamics. Now we discuss other, less explored, situations, when the collective dynamics of an ensemble is complex.

A. Partial synchrony

In case of the Kuramoto model of identical oscillators, only two situations are possible: full synchrony for attractive coupling (order parameter equal to one) and fully asynchronous state (zero order parameter) for repulsive coupling. However, this situation is not general and we can face a case, when both fully synchronous and fully asynchronous states are unstable, so that the system settles somewhere in between, at a state which is often called partial synchrony. The simplest and well-known example of partial synchrony is clustering; below we discuss several other situations where oscillators are not organized in clusters.

We exemplify partial synchrony with $N \gg 1$ mean field coupled oscillators⁶⁹:

$$\begin{aligned}\dot{x}_k &= y_k - x_k^3 + 3x_k^2 - z_k + 5 + \varepsilon(X - x_k) , \\ \dot{y}_k &= 1 - 5x_k^2 - y_k , \\ \dot{z}_k &= 0.006[4(x_k + 1.56) - z_k] ,\end{aligned}\tag{23}$$

where $k = 1, \dots, N$ and $X = N^{-1} \sum_{j=1}^N x_j$. Here individual units represent a popular Hindmarsh-Rose neuronal model⁷⁰. The quite standard parameter values used here correspond to a limit-cycle solution for the uncoupled neurons, or, in neuroscience language, to a state of periodic spiking. Fully synchronous state, $x_1 = x_2 = \dots = x_N$, $y_1 = y_2 = \dots = y_N$, $z_1 = z_2 = \dots = z_N$ is obviously a solution of the system. However, stability of this state depends on the coupling strength ε , as can be checked numerically by virtue of computation of the transversal stability via evaporation multipliers^{71,72}. This analysis, confirmed by

direct simulation, demonstrates, that there exist a critical coupling value ε_c , at which the synchronous solution loses its stability via a Hopf-like bifurcation, i.e. two complex multipliers cross the unit circle, giving birth to a new frequency. (Notice that the stability of the synchronous solution is re-gained for very large ε .) Beyond the synchrony breaking, the states of oscillators in the phase space form a thin stripe, stretched along the limit cycle; the points within this stripe slowly interchange their position, with a characteristic time of tens of periods. Roughly speaking, the average frequency of all oscillators and of the mean field is the same, but the phase shift of the oscillators with respect to the field is slowly modulated. As a result, the dynamics looks quite complicated and is possibly weakly chaotic⁶⁹.

For another example we consider a popular model of a series array of resistively shunted Josephson junctions. The junctions are coupled by virtue of a parallel RLC -load³⁶. As has been shown in cited refs., for a weak coupling and linear load, the system is equivalent to the Kuramoto model. Consider now a *nonlinear coupling*; namely, let the inductance in the RLC -circuit be nonlinear so that the magnetic flux depends on the current \dot{Q} through the RLC -load as $\Phi = L_0\dot{Q} + L_1\dot{Q}^3$. Numerical study⁷² reveals a synchrony-breaking transition: at $\varepsilon = \varepsilon_c$ the synchronous state becomes unstable; at this critical coupling value real evaporation multiplier μ becomes larger than one (in contradistinction to example (23) where the multiplier is complex). For $\varepsilon > \varepsilon_c$ the systems is in a state of partial synchrony, with the order parameter being a smooth decreasing function of ε . Furthermore, the dynamics becomes quasiperiodic: the frequency of the mean field is larger than the frequency of individual junctions, so that the latter are not locked to the field. The frequency difference grows with $\varepsilon - \varepsilon_c$.

A general theory of partially synchronous states which appear after the synchrony breaking is still missing and requires further investigations. Below we present an analytically tractable and relatively transparent example.

B. Self-organized quasiperiodic dynamics

Now we analyze the system of $N \gg 1$ Stuart-Landau oscillators:

$$\begin{aligned} \dot{a}_k = (1 + i\omega)a_k - (1 + i\alpha)|a_k|^2a_k + (\xi_1 + i\xi_2)A \\ - (\eta_1 + i\eta_2)|A|^2A, \end{aligned} \tag{24}$$

where $A = N^{-1} \sum_{j=1}^N a_j$ and $\xi_{1,2}$, $\eta_{1,2}$ are coupling parameters. This model differs from that of Refs. [64] due to the nonlinearity in coupling. In the weak coupling approximation the model with purely linear coupling⁶⁴ (i.e. with $\eta_1 = \eta_2 = 0$) yields the Kuramoto-Sakaguchi Eq. (4), while the nonlinear Eq. (24) reduces to a particular case of the following phase model⁷³:

$$\dot{\varphi} = \omega + \varepsilon \alpha(\varepsilon, R) \sin[\Theta - \varphi_k + \beta(\varepsilon, R)] \quad (25)$$

Equation (25) can be considered as an extension of the Kuramoto-Sakaguchi model. Here R is the mean field amplitude and the bifurcation parameter ε corresponds to a combination of the parameters $\xi_{1,2}$, $\eta_{1,2}$ and $\alpha(\varepsilon, R)$, $\beta(\varepsilon, R)$ are some functions. Notice that Eq. (25) appears also in a model of Stuart-Landau oscillators coupled via a common nonlinear medium^{72,73}, similarly to the Josephson junction model.

Let us consider the effect of these functions separately, starting with the case when $\beta = \text{const}$, $|\beta| < \pi/2$, i.e. the coupling is attractive, cf. Refs. [74,19]. Suppose that α is a decreasing function of ε , e.g. $\alpha(\varepsilon, R) = (1 - \varepsilon R^2)R$ (this function indeed appears for a certain combination of parameters in Eq. (24)). For $\varepsilon < \varepsilon_c = 1$ this function is positive for fully synchronous case $R = 1$ and, hence, this state is stable. For $\varepsilon > \varepsilon_c$ we have $\alpha(\varepsilon, 1) < 0$, i.e. the coupling becomes repulsive. As a result, the system stays at the border between attraction and repulsion, determined by the condition $\varepsilon R^2 = 1$, forming a *self-organized bunch state*⁷³. In this state, for general initial conditions the oscillator phases spread around the unit circle so that $R = 1/\sqrt{\varepsilon}$; this bunch is stationary in the coordinate frame, rotating with the frequency ω .

Now we analyze a more interesting case when $\alpha = R$ and $\beta(\varepsilon, R) = \beta_0 + \beta_1 \varepsilon^2 R^2$, $|\beta_0| < \pi/2$, $\beta_1 > 0$. It is easy to see that synchrony ($R = 1$) is stable if $\beta_0 + \beta_1 \varepsilon^2 < \pi/2$ and becomes unstable when ε exceeds the critical value $\varepsilon_c = \sqrt{(\pi/2 - \beta_0)/\beta_1}$. Again, the system settles at the border of stability, so that the condition $\beta_0 + \beta_1 \varepsilon^2 R^2 = \pi/2$ is fulfilled. However, in this case the dynamics exhibit a peculiar feature, also possessed by the Josephson junction model discussed above: the frequency of the mean field differs from the frequency of the individual units. Thus, the state can be characterized by two generally irrationally related frequencies, and therefore is denoted as self-organized quasiperiodicity (SOQ). Qualitatively, the emergence of quasiperiodic motion can be explained as follows. For $\varepsilon > \varepsilon_c$ the system is partially synchronous, i.e. $R < 1$ and all oscillators have different phases (notice that for

general initial conditions without clusters the phases must be different according to Eq. (7)). Hence, the instantaneous frequencies of oscillators differ and, therefore a stationary (in a rotating coordinate frame) distribution is not a solution. Quantitative analysis of SOQ dynamics with computation of the frequencies of the collective mode and of the oscillators can be found in Refs.^{72,73}. SOQ states in real-world systems have been demonstrated in experiments with electronic circuits with global nonlinear coupling⁶.

To complete the discussion of this issue we mention that similar complex states with a nonzero collective mode can appear also without desynchronization transition. For some systems, the fully synchronous and completely asynchronous states are unstable already for infinitely small coupling. A well-studied example is the van Vreeswijk model of globally coupled leaky integrate-and-fire neurons⁷⁵. Another example is the emergence of dephasing and bursting in a system of Morris-Lecar neuronal models⁷⁶; computation of the evaporation multipliers for this system shows that the synchronous state is unstable for the positive coupling range, where the effect is observed.

C. Chimeralike states in globally coupled systems

We have already mentioned a symmetry-breaking chimera state in a system of two interacting Kuramoto populations. Now we discuss emergence of a chimeralike state in a single homogeneous population. At first glance, such regimes are not allowed, because identical units subject to a common force should exhibit the same dynamics (i.e. to be either all synchronized or all desynchronized). On the other hand, there is a number of numerical observations of the mixed states, where only a fraction of the population merges to one or several clusters, while other elements remain scattered⁷⁷, see also recent studies of chimera states in linearly and nonlinearly coupled Stuart-Landau oscillators⁷⁸. The conditions for emergence of such mixed states are not yet clear. Nevertheless, we can outline one mechanism which results in splitting of the population into coherent and incoherent groups.

Identical elements subject to the same force can behave differently if they are bistable (multistable), i.e. possess at least two nontrivial attractors. Another requirement is that the mean field forcing shall be synchronizing for the units on one attractor and repulsive for the elements on the other one. Taking into account that this partially coherent and partially incoherent state shall be maintained self-consistently, we conclude that this condition

is not trivial. To illustrate this mechanism, we first consider a rather artificial but transparent example⁷⁹, where the oscillators are the non-isochronous modified Stuart-Landau systems. The modification refers to the nonlinearity: in addition to the 3rd order term $|a_k|^2 a_k$ we add also the terms $\sim |a_k|^4 a_k$, $\sim |a_k|^6 a_k$, cf. Eq. (24). As a result the systems possess two stable limit cycles with different frequencies, $\Omega_2 > \Omega_1$. The global coupling has its own dynamics, cf. Refs. [36,80], so that the mean field forces a harmonic oscillator, $\ddot{u} + \gamma\dot{u} + \eta^2 u = N^{-1} \sum_j^N \text{Re}(a_j)$, and its output \dot{u} acts on the Stuart-Landau systems. Suppose now that parameters are chosen in such a way that $\Omega_1 < \eta < \Omega_2$. Since the phase shift introduced by the harmonic oscillator in the last equation is frequency-dependent, the coupling synchronizes the large amplitude limit-cycle oscillations, but prevents synchronization of the low-amplitude ones.

Much less trivial is emergence of coherent-incoherent states in an ensemble of globally coupled oscillators with internal delayed feedback loop. Such oscillators naturally appear, e.g. in laser optics⁸¹ as well as in numerous biological applications⁸². In the simplest case the autonomous dynamics of an oscillator with a delayed feedback loop can be described by a phase model, $\dot{\varphi} = \omega + \alpha \sin(\varphi_\tau - \varphi)$, where $\varphi_\tau \equiv \varphi(t - \tau)$, τ is the delay, and α quantifies the feedback strength^{81,83}. Assuming the global coupling in the ensemble of such oscillators to be of the Kuramoto-Sakaguchi type, one writes the model as⁷⁹:

$$\dot{\varphi}_k = \omega + \alpha \sin(\varphi_{\tau,k} - \varphi_k) + \varepsilon R \sin(\Theta - \varphi_k + \beta). \quad (26)$$

Stability analysis of the fully synchronous one-cluster state yields that it is unstable for $\beta > \pi/2$. However, numerical simulation shows that for $\beta \gtrsim \pi/2$ the asynchronous state with zero mean field, $R = 0$, is also unstable (see⁷⁹ for other parameter values). Thus, the system “chooses” a state between full coherence and full incoherence, and in a range of parameters this state is reminiscent of a chimera: there is one big cluster (about 70% of the population size) and a cloud of asynchronous oscillators. The frequency of the cluster is larger than the frequency of the cloud oscillators. Noteworthy, these states appear in the case when autonomous oscillators are monostable, though close to the bistability domain (for sufficiently strong feedback, $\alpha\tau > 1$, one time-delayed unit admits multiple periodic solutions). It means that the systems become bistable due to coupling and, thus, the chimera-like state emerges due to the dynamically sustained bistability.

VII. AN IMPORTANT APPLICATION: NEUROSCIENCE

Within fifty years since Norbert Wiener hypothesized the role of the collective synchrony in the brain dynamics, importance of this phenomenon for neuroscience became highly recognized. It is now well-accepted that macroscopic neural rhythms appear due to coordination of firing and/or bursting of interconnected neuronal network and the Kuramoto model, in spite of its simplicity, became a paradigmatic and widely used setup in this field, see⁸⁴ for a review. Noteworthy, the neuroscience applications turned out to be very fruitful for the theoretical development, posing quite interesting, from the viewpoint of nonlinear dynamics, problems. In particular, in the context of complex dynamical states which are of general interest, we have several times mentioned neuronal models, e.g. the Hindmarsh-Rose one. Below, without an ambition to provide a comprehensive review, we outline several relevant issues.

For neuronal models, synchronization in a fully-connected network was observed for map-based and continuous time models, both for the regimes of periodic spiking and bursting, see e.g. Ref.⁸⁵. A bit more detailed models consist of excitatory and inhibitory neuronal subpopulations⁸⁶. Another issue is a detailed description of synaptic coupling⁸⁷, also with account of plasticity⁸⁸.

Fully connected network is certainly a rather crude approximation for a neuronal population. However, in some cases it is quite reasonable, as indicated by a numerical study of randomly coupled networks of map-based neurons⁸⁶: if every unit is connected only to 0.5% of elements in the ensemble, then synchronization properties are practically indistinguishable from the fully connected case. For many problems the assumption of random connectivity is not appropriate; in this case one has to use physiologically motivated connectivity structure. An interesting approach has been recently suggested to treat large random networks of coupled oscillators⁸⁹. To adequately perform the thermodynamic limit and preserve disorder due to randomness of connections, a heterogeneous mean-field approach has been developed in which disorder remains the same while the size of the system grows. This approach yields a description of both microscopic and global features of neuronal synchrony in the model⁸⁹. Another interesting observation relates to diluted random networks of spike-coupled neurons⁹⁰: while for weak coupling synchrony establishes quite quickly, for large coupling a very long (in fact, exponential in the network size) transient disordered state is observed,

characterized by a negative largest Lyapunov exponent (so-called “stable chaos”).

A. Control of collective synchrony

As an example of a particular application of the discussed theoretical ideas we address the problem of suppression of the collective synchrony. This problem is relevant for a medical technique, called deep brain stimulation (DBS). This technique is exploited to treat the Parkinson’s disease if it cannot be cured by medication. The DBS implies electrical stimulation of deep brain structures through the implanted micro-electrodes. The modern devices use the constant frequency (ca. 100 – 120Hz) pulse stimulation, which is typically applied around the clock. Although the exact mechanisms of DBS are not yet quite understood, it is widely used to reduce the limb tremor.

Analysis of electrical or magnetic brain activity shows that Parkinsonian tremor is associated with the pronounced spectral power at 10 to 12 Hz⁹¹. Hence, it is reasonable to hypothesize that this pathological rhythm emerges due to synchronization in a neuronal population and to consider the DBS as a desynchronization problem⁹². The main idea is then to develop efficient techniques which reliably suppress the unwanted activity with minimal stimulation. There are several directions in these studies. The first one implies open-loop stimulation with specially organized pulses⁹³ through several closely spaced sites; the main assumption is that these pulses cause formation of several symmetrically positioned clusters, so that the mean field vanishes.

Another direction in control of collective oscillation, based on the closed-loop feedback, was suggested in Refs. [94,85] and verified in experiments with electrochemical oscillators in Ref. [5]. The idea is quite transparent. Consider a globally coupled system: all elements are forced by the collective mode and synchronize due to this forcing. Suppose we measure the collective oscillation and feed it back, shifting its phase and properly amplifying it so that the feedback signal exactly compensates the mean field. In this case the oscillators become unforced and naturally desynchronize due to frequencies mismatch, internal noise, etc. Since the units desynchronize, the mean field tends to a constant, and so does the feedback signal. The constant component of the feedback signal can be easily eliminated; in this way one performs a *vanishing stimulation* control. It means that stimulation tends to zero as soon as the undesired oscillation is suppressed; this property is extremely important

for medical applications. The vanishing stimulation control can be implemented via a delayed feedback^{85,86} (see Ref. [95] for a variant with nonlinear delayed feedback which is however not vanishing) or via a phase shifting passive system without delay⁹⁶. Moreover, an adaptive strategy can be implemented⁹⁷, so that the desired state can be achieved in spite of unknown parameters of the system to be controlled. Notice that adjusting the phase shift introduced by the feedback loop one can, depending on the goal, suppress or enhance the collective synchrony.

VIII. SURPRISES AND OUTLOOK

In the first years after development of ideas of synchronization in large ensembles, especially after Y. Kuramoto constructed his self-consistent theory, it looked like the very phenomenon of synchronization transition is rather simple and universal. In this paper we tried to show how in fact nontrivial is even the simplest setup: features like partial integrability, existence of an exact low-dimensional manifold, nontrivial transition scenaria even for unimodal distributions of frequencies, chimera states, self-organized quasiperiodicity occur already in the simplest case of sine-coupled phase oscillators. For generalizations of this model one observes a plethora of dynamical phenomena which is still far from being exhausted. Furthermore, addition of complexity to the basic model, e.g. by consideration of coupled oscillators on networks, results in further growth of the diversity of possible regimes.

It seems to us that in the nearest future we will experience spreading of the synchronization theory far beyond its initial scope of nonlinear dissipative dynamical systems, e.g. to cover quantum objects⁹⁸. On the experimental level, advanced methods of oscillators' control and of data analysis will possibly reveal microscopic details of nontrivial synchronization patterns. On the other hand, growing interest of mathematicians to these problems indicates that the field may become a part of mathematical physics as well.

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